X-fluid and viscous fluid in D-dimensional anisotropic integrable cosmology

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Abstract

D-dimensional cosmological model describing the evolution of a perfect fluid with negative pressure (x-fluid) and a fluid possessing both shear and bulk viscosity in n Ricci-flat spaces is investigated. The second equations of state are chosen in some special form of metric dependence of the shear and bulk viscosity coefficients. The equations of motion are integrated and the dynamical properties of the exact solutions are studied. It is shown the possibility to resolve the cosmic coincidence problem when the x-fluid plays role of quintessence and the viscous fluid is used as cold dark matter.

1 Introduction

The most probable candidate for the so called quintessence matter responsible for the current phase of the accelerated expansion of the Universe is a Λ term o more generally an exotic x-fluid - perfect fluid with negative pressure satisfying a linear barotropic equation of state (see, for instance, [1] and refs. therein). To describe the present stage of evolution this x-fluid is to be added to the normal matter, which mainly consist of cold dark matter. For instance, ΛCDM model [2],[3] describes the flat Friedman-Robertson-Walker (FRW) Universe filled with a mixture of quintessence represented by the cosmological constant Λ and cold dark matter in the form of pressureless perfect fluid (dust).

However, the flat FRW cosmologies with a x-fluid and a normal perfect fluid are not free from some difficulties. One of them is the so called coincidence problem [4]: to explain why the quintessence density and the normal matter density are comparable today, one has to tune their initial ratio very carefully. The problem may be ameliorated by replacement the x-fluid by the so called quintessence scalar field - homogeneous scalar field Q slowly rolling down with some potential $V(Q)$ [5],[6]. For instance, the potential $V(Q) = M^{4+\alpha}Q^{-\alpha}$, $\alpha > 0$, leads to the so called "tracker" solution, which "attracts" solutions to the equations of motion before the present stage for very wide range of initial conditions. Another way for resolving the problem was proposed in [7]. The idea is to use a fluid with bulk viscosity in combination with a quintessence scalar field.

The aim of this paper is to show that the cosmic coincidence problem may be resolved by using a x-fluid as the quintessence and a viscous fluid as the normal matter. In section 2 we describe the general model and get basic equations. To integrate the equations of motion we choose the so called "second equations of state", which provide us with the dependence of the shear and bulk viscosity coefficients on time, in some special form of their metric

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dependence. The exact solutions for both isotropic and anisotropic case are obtained in section 3, where their dynamical properties are discussed.

2 The general model

We assume the following metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\gamma(t)}\mathrm{d}t^2 + \sum_{i=1}^{n} \exp[2x^i(t)]\mathrm{d}s_i^2, \tag{2.1}$$

on the D-dimensional space-time manifold

$$\mathbb{M} = \mathbb{R} \times M_1 \times \ldots \times M_n, \tag{2.2}$$

where $\mathrm{d}s_i^2$ is the metric of the Ricci-flat factor space M_i of dimension d_i, $\gamma(t)$ and $x^i(t)$ are scalar functions of the cosmic time t. $a_i \equiv \exp[x^i]$ is the scale factor of the space M_i and the function $\gamma(t)$ determines a time gauge. The synchronous time t_s is defined by the equation $\mathrm{d}t_s = \exp[\gamma(t)]\mathrm{d}t$.

We consider a source of gravitational field in the form of 2-component cosmic fluid. The first one is a perfect fluid with a density $\rho^{(1)}(t)$ and a pressure $p^{(1)}(t)$. The second component supposed to be a viscous fluid. It is characterized by a density $\rho^{(2)}(t)$, a pressure $p^{(2)}(t)$, a bulk viscosity coefficient $\zeta(t)$ and a shear viscosity coefficient $\eta(t)$. The overall energy-momentum tensor of the cosmic fluid reads

$$T^\mu_\nu = \left(\rho^{(1)} + \rho^{(2)}\right) u^\mu u_\nu + \left(p^{(1)} + p^{(2)} - \zeta\theta\right) P^\mu_\nu - 2\eta\sigma^\mu_\nu, \tag{2.3}$$

where u^μ is the D-dimensional velocity of the fluid, $\theta = u^\mu_{;\mu}$ denotes the scalar expansion, $P^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$ is the projector on the $(D-1)$-dimensional space orthogonal to u^μ, $\sigma^\mu_\nu = \frac{1}{2}\left(u_{\alpha;\beta} + u_{\beta;\alpha}\right) P^{\alpha\mu} P^\beta_\nu - (D-1)^{-1}\theta P^\mu_\nu$ is the traceless shear tensor and $\mu, \nu = 0, 1, \ldots, D-1$.

By assuming the comoving observer condition $u^\mu = \delta^\mu_0 \mathrm{e}^{-\gamma(t)}$, the overall energy-momentum tensor may be written as

$$T^\mu_\nu = T^{\mu(1)}_\nu + T^{\mu(2)}_\nu, \tag{2.4}$$

where

$$\left(T^{\mu(1)}_\nu\right) = \mathrm{diag}(-\rho^{(1)}, p^{(1)}, \ldots, p^{(1)}), \tag{2.5}$$

$$\left(T^{\mu(2)}_\nu\right) = \mathrm{diag}(-\rho^{(2)}, \tilde{p}^{(2)}_1 \delta^{k_1}_{l_1}, \ldots, \tilde{p}^{(2)}_n \delta^{k_n}_{l_n}), \tag{2.6}$$

$k_i, l_i = 1, \ldots, d_i$ for $i = 1, \ldots, n$. Here $\tilde{p}^{(2)}_i$ denotes the effective pressure including the dissipative contribution of the viscous fluid in the factor space described by the manifold M_i. It reads

$$\tilde{p}^{(2)}_i = p^{(2)} - \mathrm{e}^{-\gamma}\left[\zeta\dot{\gamma}_0 + 2\eta\left(\dot{x}^i - \frac{\dot{\gamma}_0}{D-1}\right)\right], \tag{2.7}$$

where

$$\gamma_0 = \sum_{i=1}^{n} d_i x^i. \tag{2.8}$$

Furthermore, we assume that the barotropic equations of state hold

$$p^{(\alpha)} = \left(1 - h^{(\alpha)}\right)\rho^{(\alpha)}, \ \alpha = 1, 2, \tag{2.9}$$

where the $h^{(\alpha)]}$ are constants such that $h^{(1)} \neq h^{(2)}$.

The Einstein equations $R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = \kappa^2 T^\mu_\nu$, where κ^2 is the gravitational constant, can be written as $R^\mu_\nu = \kappa^2(T^\mu_\nu - [T/(D-2)]\delta^\mu_\nu)$. Further, the equations $R^0_0 - \frac{1}{2}\delta^0_0 R = \kappa^2 T^0_0$ and $R^a_b = \kappa^2(T^a_b - [T/(D-2)]\delta^a_b)$, where $a, b = 1, \ldots, D$, give the following equations of motion

$$\begin{aligned}
\dot{\gamma}_0^2 - \sum_{i=1}^n d_i(\dot{x}^i)^2 &= 2\kappa^2 \mathrm{e}^{2\gamma}\left(\rho^{(1)} + \rho^{(2)}\right), &(2.10)\\
\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i &= \kappa^2 \mathrm{e}^{\gamma}\left[\frac{\mathrm{e}^\gamma}{D-2}\sum_{\alpha=1}^2 h^{(\alpha)}\rho^{(\alpha)} + \frac{\zeta}{D-2}\dot{\gamma}_0 - 2\eta\left(\dot{x}^i - \frac{\dot{\gamma}_0}{D-1}\right)\right], &(2.11)
\end{aligned}$$

$i = 1, \ldots, n$.

The energy conservation law $\bigtriangledown_\mu T_0^{\mu(2)} = 0$ for a viscous fluid described by a tensor given by equation (2.6) reads

$$\dot{\rho}^{(2)} + \sum_{i=1}^n d_i \dot{x}^i \left(\rho^{(2)} + \tilde{p}_i^{(2)}\right) = 0. \tag{2.12}$$

Owing to the constraint $\bigtriangledown_\mu T^\mu_0 = 0$ for the overall energy-momentum tensor given by equation (2.3) the similar energy conservation law is valid for the perfect fluid

$$\dot{\rho}^{(1)} + \left(\rho^{(1)} + p^{(1)}\right)\sum_{i=1}^n d_i \dot{x}^i = 0. \tag{2.13}$$

Taking into account equation (2.9), ones easily integrates equation (2.13). The result is

$$\rho^{(1)} = Ae^{(h^{(1)}-2)\gamma_0}, \tag{2.14}$$

where A is a positive constant. We note that the contribution of the perfect fluid component with $h^{(2)} = 2$ to the overall energy momentum tensor is equivalent to the presence of Λ-term with $\Lambda = \kappa^2 A$.

Furthermore, by using equations (2.10) and (2.14) the presence of the densities $\rho^{(1)}$ and $\rho^{(2)}$ in equations (2.11) can be cancelled. Thus, we obtain the main governing set of equations

$$\begin{aligned}
\ddot{x}^i + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^i &= \kappa^2 A\frac{h^{(1)} - h^{(2)}}{D-2}\mathrm{e}^{[h^{(1)}-2]\gamma_0 + 2\gamma} + \frac{h^{(2)}}{2(D-2)}\left(\dot{\gamma}_0^2 - \sum_{i=1}^n d_i(\dot{x}^i)^2\right)\\
&+ \kappa^2 \mathrm{e}^\gamma\left[\frac{\zeta}{D-2}\dot{\gamma}_0 - 2\eta\left(\dot{x}^i - \frac{\dot{\gamma}_0}{D-1}\right)\right]. &(2.15)
\end{aligned}$$

We use an integration procedure which is based on the n-dimensional Minkowsky-like geometry. Let $\mathrm{I\!R}^n$ be the real vector space and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis in $\mathrm{I\!R}^n$ (i.e. $\mathbf{e}_1 = (1, 0, \ldots, 0)$ etc). Let us define a symmetrical bilinear form $\langle , \rangle$ on $\mathrm{I\!R}^n$ by

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}d_j - d_i d_j. \tag{2.16}$$

Such a form is non-degenerate and has the pseudo-Euclidean signature $(-,+,\ldots,+)$ [8] . With this in mind, a vector $\mathbf{y} \in \mathbb{R}^n$ is timelike, spacelike or isotropic respectively, if $\langle \mathbf{y}, \mathbf{y} \rangle$ takes negative, positive or null values respectively and two vectors $\mathbf{y}$ and $\mathbf{z}$ are orthogonal if $\langle \mathbf{y}, \mathbf{z} \rangle = 0$. Hereafter, we use the following vectors

$$\mathbf{x} = x^1(t)\mathbf{e}_1 + \ldots + x^n(t)\mathbf{e}_n, \tag{2.17}$$

$$\mathbf{u} = u^1\mathbf{e}_1 + \ldots + u^n\mathbf{e}_n, \ u^i = \frac{-1}{D-2}, \ u_i = d_i, \tag{2.18}$$

where the covariant coordinates u_i of the vector $\mathbf{u}$ are introduced by the usual way. Moreover, we obtain

$$\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle = \sum_{i=1}^{n} d_i (\dot{x}^i)^2 - \dot{\gamma}_0^2, \ \langle \mathbf{u}, \mathbf{x} \rangle = \gamma_0, \ \langle \mathbf{u}, \mathbf{u} \rangle = -\frac{D-1}{D-2}. \tag{2.19}$$

Thus, using equations (2.17)-(2.19) we rewrite the main governing set of equations in the following vector form

$$\begin{aligned} \ddot{\mathbf{x}} + (\dot{\gamma}_0 - \dot{\gamma})\dot{\mathbf{x}} &= \left[\kappa^2 A \left(h^{(2)} - h^{(1)}\right) \mathrm{e}^{[h^{(1)}-2]\gamma_0 + 2\gamma} + \frac{h^{(2)}}{2}\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle\right] \mathbf{u} \\ &- \kappa^2 \mathrm{e}^{\gamma} \left[\left(\zeta + \frac{D-2}{D-1} 2\eta\right) \dot{\gamma}_0 \mathbf{u} + 2\eta \dot{\mathbf{x}}\right]. \end{aligned} \tag{2.20}$$

Moreover, the density $\rho^{(2)}$ can be expressed as

$$\rho^{(2)} = -\frac{\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle}{2\kappa^2} \mathrm{e}^{-2\gamma} - \rho^{(1)}. \tag{2.21}$$

Now we summarize thermodynamics principles. The first law of thermodynamics applied to the viscous fluid reads

$$T \mathrm{d}S = \mathrm{d}\left(\rho^{(2)} V\right) + p^{(2)} \mathrm{d}V, \tag{2.22}$$

where V stands for a fluid volume in the whole space $M_1 \times \ldots \times M_n$, S is an entropy in the volume V and T is a temperature of the viscous fluid. By assuming that the baryon particle number N_{B} in the volume V is conserved, equation (2.22) transforms to

$$nT\dot{s} = \dot{\rho}^{(2)} + \left(\rho^{(2)} + p^{(2)}\right) \sum_{i=1}^{n} d_i \dot{x}^i \tag{2.23}$$

where $s = S/N_{\mathrm{B}}$ and $n = N_{\mathrm{B}}/V$ stands for the entropy per baryon and the baryon number density. The comparison between equations (2.12) and (2.23) gives the variation rate of entropy per baryon

$$nT\dot{s} = \sum_{i=1}^{n} d_i \dot{x}^i \left(p^{(2)} - \tilde{p}_i^{(2)}\right) = \mathrm{e}^{-\gamma} \left[\left(\zeta + \frac{D-2}{D-1} 2\eta\right) \dot{\gamma}_0^2 + 2\eta \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle\right]. \tag{2.24}$$

3 Exact solutions

To integrate equation (2.20) one needs a second set equations of state, involving the bulk viscosity coefficient ζ and the shear viscosity coefficient η. Herein, we suppose

$$\zeta = \frac{\zeta_0}{\kappa^2}\frac{D-2}{D-1}\dot{\gamma}_0 \mathrm{e}^{-\gamma},\ \eta = \frac{\eta_0}{2\kappa^2}\dot{\gamma}_0 \mathrm{e}^{-\gamma}, \tag{3.1}$$

where $\zeta_0 \geq 0$ and η_0 are constants. When the cosmological model is isotropic, i.e. $\dot{x}^i = \dot{\gamma}_0/(D-1)$, $i = 1, \ldots, n$, and the shear viscosity is not significant, the expression $H \equiv \dot{\gamma}_0 \mathrm{e}^{-\gamma}/(D-1)$ is the Hubble parameter.Then we get from equation (3.1): $\zeta = \frac{D-2}{\kappa^2}\zeta_0 H$, i.e. the bulk viscosity coefficient is linear proportional to the Hubble parameter. Such kind of the second equation of state describes the so called "linear dissipative regime" in the FRW world model (see, for instance, [7]). As we study an anisotropic cosmological model, we must involve a shear viscosity as well as a bulk one. So, we propose equations (3.1) for the anisotropic model as a simplest generalization to the of the second equation of state describing the linear dissipative regime.

In order to integrate equation (2.20), we use the orthogonal basis

$$\frac{\mathbf{u}}{\langle \mathbf{u}, \mathbf{u} \rangle}, \mathbf{f}_2, \ldots, \mathbf{f}_n \in \mathbb{R}^n, \tag{3.2}$$

where the vector $\mathbf{u}$ was introduced by equation (2.18) The orthogonality property reads

$$\langle \mathbf{u}, \mathbf{f}_j \rangle = 0,\ \langle \mathbf{f}_j, \mathbf{f}_k \rangle = \delta_{jk}, \quad (j, k = 2, \ldots, n). \tag{3.3}$$

Let us note that the basis vectors $\mathbf{f}_2, \ldots, \mathbf{f}_n$ are space-like, since they are orthogonal to the time-like vector $\mathbf{u}$. The vector $\mathbf{x} \in \mathbb{R}^n$ decomposes as follows

$$\mathbf{x} = \gamma_0 \frac{\mathbf{u}}{\langle \mathbf{u}, \mathbf{u} \rangle} + \sum_{j=2}^{n} \langle \mathbf{x}, \mathbf{f}_j \rangle \mathbf{f}_j. \tag{3.4}$$

Hence, under the above assumptions equation (2.20) reads in the terms of coordinates in such basis as follows

$$\begin{aligned} \ddot{\gamma}_0 + \left[\left(1 - \tfrac{h^{(2)}}{2} - \zeta_0\right)\dot{\gamma}_0 - \dot{\gamma}\right]\dot{\gamma}_0 \quad &= \\ \frac{D-1}{D-2} \quad \left[\kappa^2 A \left(h^{(1)} - h^{(2)}\right) \mathrm{e}^{[h^{(1)}-2]\gamma_0 + 2\gamma} - \tfrac{h^{(2)}}{2} \textstyle\sum_{j=2}^{n} \langle \dot{\mathbf{x}}, \mathbf{f}_j \rangle^2\right] \quad &, \qquad (3.5) \\ \langle \ddot{\mathbf{x}}, \mathbf{f}_j \rangle + \left[(1+\eta_0)\dot{\gamma}_0 - \dot{\gamma}\right] \langle \dot{\mathbf{x}}, \mathbf{f}_j \rangle \quad &= 0 \quad (j = 2, \ldots, n). \qquad (3.6) \end{aligned}$$

The integration of equation (3.6) gives

$$\langle \dot{\mathbf{x}}, \mathbf{f}_j \rangle = p^j \mathrm{e}^{\gamma - (1+\eta_0)\gamma_0}, \tag{3.7}$$

where p^j is an arbitrary constant.

Further we determine the time gauge by

$$\gamma = k\gamma_0, \tag{3.8}$$

where k is a constant.

By substituting the functions $\langle \dot{\mathbf{x}}, \mathbf{f}_j \rangle$ into equation (3.5) we obtain the following integrable by quadrature ordinary differential equation

$$\ddot{\gamma}_0 - \left(k - 1 + \frac{h^{(2)}}{2} + \zeta_0\right) \dot{\gamma}_0^2 =$$
$$\frac{D-2}{D-1} \mathrm{e}^{2(k-1)\gamma_0} \left[\kappa^2 A \left(h^{(1)} - h^{(2)}\right) \mathrm{e}^{h^{(1)}\gamma_0} - \frac{h^{(2)}}{2} \sum_{j=2}^{n} \left(p^j\right)^2 \mathrm{e}^{-2\eta_0\gamma_0}\right]. \tag{3.9}$$

In what follows we accept the agreement

$$\mathrm{d}t > 0, \tag{3.10}$$

i.e. the cosmic time increases during the evolution.

It should be noted that the solutions of equation (3.9) corresponding to different sets of the parameters $h^{(1)}, h^{(2)}, \zeta_0$ and η_0 may lead to nonsatisfactory from the physical viewpoint cosmological evolutions. Further, we study only ones, which satisfy the following *consistency condition*: neither the density $\rho^{(2)}(t)$ nor the variation rate of entropy $\dot{s}(t)$ have negative values on any time interval.

3.1 The isotropic model

The *isotropic model* is described by the metric given by equation (2.1) with

$$a \equiv \mathrm{e}^{x^i} = \mathrm{e}^{\gamma_0/(D-1)}, \ i = 1, \ldots, n. \tag{3.11}$$

The scale factor a of the whole isotropically evolving space $M_1 \times \ldots \times M_n$ can be obtained by integration of equation (3.9) with

$$p^j = 0, \ j = 2, \ldots, n. \tag{3.12}$$

It can be proven that the above mentioned *consistency condition* leads to the following constraint

$$h^{(1)} - h^{(2)} - 2\zeta_0 > 0. \tag{3.13}$$

At first we present the special solution to equation (3.9). The special solution describes the asymptotical behaviour of all solutions at late time. It is the steady state solution

$$a \sim \exp\left[\sqrt{\frac{2\Lambda(1+\varepsilon_0)}{(D-1)(D-2)}} t_s\right], \ \rho^{(1)} = A \equiv \Lambda/\kappa^2, \tag{3.14}$$

for $h^{(2)} = 2$ and shows a power-law behaviour

$$a \sim t_s^{2/[(2-h^{(1)})(D-1)]}, \ \rho^{(1)} \sim t_s^{-2} \tag{3.15}$$

for $h^{(1)} \neq 2$, where t_s is the synchronous time. Moreover, we present the deceleration parameter

$$q \equiv -\frac{a(\ddot{a} - \dot{\gamma}\dot{a})}{\dot{a}^2} = -1 + \frac{D-1}{2}\left(2 - h^{(1)}\right), \tag{3.16}$$

the density ratio

$$\rho^{(2)}/\rho^{(1)} = \varepsilon_0 \equiv \frac{2\zeta_0}{h^{(1)} - h^{(2)} - 2\zeta_0} \tag{3.17}$$

and the overall pressure

$$p^{(1)} + \tilde{p}^{(2)} = \left(1 - h^{(1)}\right)(1+\varepsilon_0)\rho^{(1)}. \tag{3.18}$$

The variation rate of entropy is positive (if $\zeta_0 > 0$) and $nT\dot{s} \sim \rho^{(1)}$.

To obtain the general solution to equation (3.9) in the isotropic case we suppose that the parameter k specifying the time gauge by equation (3.8) is

$$k = 1 - h^{(1)}/2. \tag{3.19}$$

(We note that the time t becomes synchronous if the perfect fluid component appears as Λ-term, i.e. $h^{(1)} = 2$). This yields that equation (3.9) under the condition given by equation (3.12) is integrable by elementary methods. The result is

$$a = a_0 \left(\frac{\tau^2}{1-\tau^2}\right)^{1/[(D-1)(h^{(1)}-h^{(2)}-2\zeta_0)]}, \tag{3.20}$$

$$\rho^{(1)} = A a^{(D-1)[h^{(1)}-2]}, \tag{3.21}$$

$$q = -1 + \frac{D-1}{2}\left[2 - h^{(1)} + \left(h^{(1)} - h^{(2)} - 2\zeta_0\right)(1-\tau^2)\right], \tag{3.22}$$

$$\rho^{(2)}/\rho^{(1)} = (1+\varepsilon_0)/\tau^2 - 1, \tag{3.23}$$

$$p^{(1)} + \tilde{p}^{(2)} = (1+\varepsilon_0)\left[\left(1 - h^{(2)} - 2\zeta_0\right)\frac{1-\tau^2}{\tau^2} + 1 - h^{(1)}\right]\rho^{(1)}, \tag{3.24}$$

where we introduced the following variable

$$\tau = \tanh\left[\left(h^{(1)} - h^{(2)}\right)\sqrt{\frac{\kappa^2(D-1)A}{2(D-2)(1+\varepsilon_0)}}(t-t_0)\right],\ t > t_0. \tag{3.25}$$

Now we analyze the role of viscosity in this isotropic 2-component model using the obtained exact solution. The main features of the model are the following. Under the weak energy condition ($\rho^{(1)} + \rho^{(2)} \geq 0,\ \rho^{(1)} + \rho^{(2)} + p^{(1)} + \tilde{p}^{(2)} \geq 0$), which leads to the following restriction on the parameters $2 \geq h^{(1)} > h^{(2)} + 2\zeta_0$, the Universe expands eternally from the initial singularity. Near the singularity we obtain in the main order $a \sim t_s^{2/[(2-h^{(2)}-2\zeta_0)(D-1)]}$ There is only nonsingular solution given by equation (3.14).

All solutions describes the accelerated expansion at least on the late phase of evolution if $h^{(1)} > 2(D-2)/(D-1)$. The solutions given by equation (3.20) describe the period of decelerated expansion if $h^{(2)} + 2\zeta_0 < 2(D-2)/(D-1)$. We note that the cosmic deceleration phase is important within mechanism of the clumping of matter into galaxies (see, for instance, [2]). Under the assumption $2 \geq h^{(1)} > 2(D-2)/(D-1) > h^{(2)} + 2\zeta_0$ the decelerated expansion takes place during the time interval (t_0, t^*), where t^* is defined by

$$t^* - t_0 = \frac{\operatorname{arcosh}\sqrt{1 + \frac{2(D-2)/(D-1) - h^{(2)} - 2\zeta_0}{h^{(1)} - 2(D-2)/(D-1)}}}{\sqrt{\kappa^2 \frac{A(D-1)}{2(D-2)}\left(h^{(1)} - h^{(2)}\right)\left(h^{(1)} - 2\frac{D-2}{D-1}\right)\left[1 + \frac{2(D-2)/(D-1) - h^{(2)} - 2\zeta_0}{h^{(1)} - 2(D-2)/(D-1)}\right]}}. \tag{3.26}$$

Equation (3.26) shows that introducing of the bulk viscosity reduces the phase of the decelerated expansion.

The density ratio given by equations (3.17) and (3.23) exhibits the following property

$$\lim_{t\to+\infty}\rho^{(2)}/\rho^{(1)} = \varepsilon_0 \equiv \frac{2\zeta_0}{h^{(1)} - h^{(2)} - 2\zeta_0}. \tag{3.27}$$

So, the bulk viscosity allow to resolve the coincidence problem which appears in the unviscous 2-component model.

3.2 The anisotropic model

Now we consider the general anisotropic behaviour of the model. Once integrating equation (3.9) we get the first integral of the form $\dot{\gamma}_0 = F(\gamma_0, C)$, where F is some function and C is an arbitrary constant. Substituting $\dot{\gamma}_0$ and the functions $\langle \dot{\mathbf{x}}, \mathbf{f}_j \rangle$ given by equation (3.7) into equations (2.21) and (2.24) we express $\rho^{(2)}$ and $nT\dot{s}$ via γ_0. The subsequent analysis of the expressions for γ_0, $\rho^{(2)}$ and $nT\dot{s}$ shows the presence of solutions with physically nonsatisfactory behaviour near the initial singularity. To exclude such solutions we put the following restrictions on the parameters

$$2 \geq h^{(1)} > h^{(2)} + 2\zeta_0 \geq -2\eta_0 \geq 2\zeta_0,\ -2\eta_0 \geq h^{(2)} \geq 0. \tag{3.28}$$

These restrictions guarantee the following properties for all solutions: the density $\rho^{(2)}$ and the variation rate of entropy $\dot{s}$ are positive during the evolution, which starts at the initial singularity of the Kasner type and proceeds eternally to the subsequent isotropic expansion; the density ratio $\rho^{(2)}/\rho^{(1)}$ tends to the nonzero constant as $t \to +\infty$ (see equation (3.27)).

To study these behaviour in detail let us obtain the exact solution. We note that the equation $\dot{\gamma}_0 = F(\gamma_0, C)$ is integrable by quadrature for arbitrary parameters $h^{(1)}, h^{(2)}, \zeta_0$ and η_0. To express the exact solution in elementary functions we put the following relation on the parameters

$$h^{(1)} - 2h^{(2)} - 2\eta_0 - 4\zeta_0 = 0. \tag{3.29}$$

For the nonviscous model ($\eta_0 = \zeta_0 = 0$) the relation reads $h^{(1)} = 2h^{(2)}$. The latter corresponds, for instance, to the so called ΛCDM cosmological model with Λ-term ($h^{(1)} = 2$) and dust ($h^{(2)} = 1$). We remind that the parameters obey the inequalities given by formula (3.28). Comparing equation (3.29) and formula (3.28), one gets $h^{(2)} > 0$.

Now we start the integration procedure in the time gauge defined by equations (3.8) and (3.19) . The integration of equation (3.9) gives

$$\mathrm{e}^{\beta\gamma_0(t)} = C_0 \frac{\tau(\sin\alpha + \tau\cos\alpha)}{1 - \tau^2}, \tag{3.30}$$

where we cancelled the constant $\sum_{j=2}^{n}(p^j)^2$ by introducing the following constants

$$C_0 \geq \sqrt{\frac{2h^{(2)}}{\kappa^2 A(h^{(1)} - h^{(2)})}\sum_{j=2}^{n}(p^j)^2},\ \ \alpha = \arcsin\frac{\sqrt{\frac{2h^{(2)}}{\kappa^2 A(h^{(1)}-h^{(2)})}\sum_{j=2}^{n}(p^j)^2}}{C_0} \in [0, \pi/2], \tag{3.31}$$

the parameter β is defined as follows

$$\beta = h^{(1)} - h^{(2)} - 2\zeta_0 > 0. \tag{3.32}$$

The variable τ was introduced by equation (3.25). Substituting equation (3.30) into equation (3.7) and taking into account equations (3.8) and (3.19), we obtain by integration

$$\langle \mathbf{x}, \mathbf{f}_j \rangle = \sqrt{\frac{(D-2)\beta}{(D-1)h^{(2)} \sum_{j=2}^{n} (p^j)^2}} p^j \ln \left[\frac{\tau}{\sin\alpha + \tau\cos\alpha} \right]^{1/\beta} + q^j, \tag{3.33}$$

where q^j are arbitrary constants. Substituting equations (3.30) and (3.33) into the decomposition given by equation (3.4), we get

$$\mathbf{x} = \ln \left[C_0 \frac{(\sin\alpha + \tau\cos\alpha)^2}{1-\tau^2} \right]^{1/\beta} \frac{\mathbf{u}}{\langle \mathbf{u}, \mathbf{u} \rangle} + \ln \left[\frac{\tau}{\tau(\sin\alpha + \tau\cos\alpha)} \right]^{1/\beta} \mathbf{r} + \mathbf{s}, \tag{3.34}$$

where the vectors $\mathbf{r} \in \mathbb{R}^n$ and $\mathbf{s} \in \mathbb{R}^n$ are defined as follows

$$\mathbf{r} \equiv \sum_{i=1}^{n} r^i \mathbf{e}_i = \sqrt{\frac{(D-2)\beta}{(D-1)h^{(2)} \sum_{j=2}^{n} (p^j)^2}} \sum_{j=2}^{n} p^j \mathbf{f}_j + \frac{\mathbf{u}}{\langle \mathbf{u}, \mathbf{u} \rangle}, \quad \mathbf{s} \equiv \sum_{i=1}^{n} s^i \mathbf{e}_i = \sum_{j=2}^{n} q^j \mathbf{f}_j. \tag{3.35}$$

Owing to the orthogonality property given by equation (3.3) the coordinates r^i and s^i of these vectors in the canonical basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ satisfy the following constraints

$$\sum_{i=1}^{n} d_i r^i = \langle \mathbf{u}, \mathbf{r} \rangle = 1, \ \sum_{i=1}^{n} d_i \left(r^i \right)^2 = \langle \mathbf{r}, \mathbf{r} \rangle - \sum_{i,j=1}^{n} d_i d_j r^i r^j = \frac{(D-2)\beta}{(D-1)h^{(2)}} + \frac{1}{D-1}, \tag{3.36}$$

$$\sum_{i=1}^{n} d_i s^i = \langle \mathbf{u}, \mathbf{s} \rangle = 0. \tag{3.37}$$

The constants r^i may be called Kasner-like parameters, because of the existence of the constraints given by equation (3.36). We remind, that the coordinates of the vector x in the canonical basis are the logarithms of the scale factors $a_i \equiv \exp[x^i]$.

Finally, we present the exact solution:

$$\begin{aligned} \mathrm{d}s^2 &= - \left[C_0 \frac{\tau(\sin\alpha + \tau\cos\alpha)}{1-\tau^2} \right]^{(2-h^{(2)})/\beta} \mathrm{d}t^2 + \left[C_0 \frac{(\sin\alpha + \tau\cos\alpha)^2}{1-\tau^2} \right]^{2/(D-1)\beta} \times \\ &\times \sum_{i=1}^{n} \left[\frac{\tau}{\sin\alpha + \tau\cos\alpha} \right]^{2r^i/\beta} \mathrm{e}^{2s^i} \mathrm{d}s_i^2, \end{aligned} \tag{3.38}$$

$$\rho^{(1)} = A \left[C_0 \frac{\tau(\sin\alpha + \tau\cos\alpha)}{1-\tau^2} \right]^{-(2-h^{(2)})/\beta}, \tag{3.39}$$

$$\rho^{(2)}/\rho^{(1)} = (1+\varepsilon_0) F_1^2(\tau) \left[1 - \frac{\beta}{h^{(2)}} F_2^2(\tau) \right] - 1, \tag{3.40}$$

$$nT\dot{s} = \sqrt{8\kappa^2 A \frac{D-1}{D-2}(1+\varepsilon_o)^3} F_1^3(\tau) \left[\zeta_0 + \frac{\eta_0 \beta}{h^{(2)}} F_2^2(\tau)\right] \rho^{(1)}, \tag{3.41}$$

where

$$F_1(\tau) = \frac{\frac{1}{2}(1+\tau^2)\sin\alpha + \tau\cos\alpha}{\tau(\sin\alpha + \tau\cos\alpha)}, \ F_2(\tau) = \frac{(1-\tau^2)\sin\alpha}{(1+\tau^2)\sin\alpha + 2\tau\cos\alpha}. \tag{3.42}$$

The solution has the following integration constants: $C_0 > 0$, $\alpha \in [0, \pi/2]$, t_0, $r^1, \ldots, r^n$, $s^1, \ldots, s^n$. The constants r^i and s^i satisfy the constraints given by equations (3.36) and (3.37). Then the number of free integration constants is $2n$ as required. The limit for $\alpha \to +0$ of this exact solution is the isotropic solution obtained in section 3.2.

Before we start the studying of the obtained exact solution near the initial singularity let us remind the multi-dimensional generalization of the well-known *Kasner solution* [9]. It reads (for the synchronous time t_s) as follows

$$\mathrm{d}s^2 = -\mathrm{d}t_s^2 + \sum_{i=1}^{n} A_i t_s^{2\varepsilon^i} \mathrm{d}s_i^2. \tag{3.43}$$

Such a metric describes the evolution of a vacuum model under consideration. The Kasner parameters ε^i satisfy the constraints

$$\sum_{i=1}^{n} d_i \varepsilon^i = 1, \quad \sum_{i=1}^{n} d_i \left(\varepsilon^i\right)^2 = 1. \tag{3.44}$$

The generalized Kasner solution describes the contraction of some spaces from the set $M_1, \ldots, M_n$ and the expansion for the other ones. According to equation (3.44), the number of either contracting or expanding spaces depends on n (the total number of spaces) and $d_{i=1,n}$ (their dimensions).

We note that the constraints given by equation (3.36) for r^i coincide with these constraints for ε^i when $\zeta_0 = \eta_0 = 0$ and $h^{(1)} = 2h^{(2)}$, i.e. the Kasner-like parameters r^i become exactly Kasner parameters ε^i in the absence of viscosity. Therefore, if the parameters ζ_0 and η_0 are small enough, then the model describes a behaviour of the Kasner type as $\tau \to +0$, i.e. towards to the initial singularity. However, too strong viscosity suppress a behaviour of the Kasner type. It can be shown that if the parameters $\zeta_0 = \eta_0 = 0$ are large enough, then the model describes expansion of all factor spaces $M_1, \ldots, M_n$ near the initial singularity.

One can prove that the final stage of the evolution ($\tau \to 1-0$) exhibits the isotropic expansion. The asymptotical behaviour of the model for $\tau \to 1-0$ is described by the exact solution given by equation (3.14)-(3.18).

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